

Asymptotics for stochastic reaction-diffusion equation driven by subordinate Brownian motions

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Abstract

We study the ergodicity of stochastic reaction-diffusion equation driven by subordinate Brownian motions. After establishing the strong Feller property and irreducibility of the system, we prove the tightness of the solution's law. These properties imply that this stochastic system admits a unique invariant measure according to Doob's and Krylov-Bogolyubov's theories. Furthermore, we establish a large deviation principle for the occupation measure of this system by a hyper-exponential recurrence criterion. It is well known that S(P)DEs driven by α -stable type noises do not satisfy Freidlin-Wentzell type large deviation, our result gives an example that strong dissipation overcomes heavy tailed noises to produce a Donsker-Varadhan type large deviation as time tends to infinity.

Keywords: Stochastic reaction-diffusion equation; Subordinate Brownian motions; Large deviation principle (LDP); Occupation measure.

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1 Introduction

Consider a stochastic reaction-diffusion equation driven by subordinate Brownian motion on torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ as follows:

$$dX - \partial_\xi^2 X dt - (X - X^3)dt = Q_\beta dL_t, \quad (1.1)$$

where $X : [0, +\infty) \times \mathbb{T} \times \Omega \rightarrow \mathbb{R}$ and L_t is a subordinate Brownian motion. More details about this equation will be given in the next section. Sometimes the equation (1.1) is also called stochastic Allen-Cahn equation or real Ginzburg-Landau equation. Recently, the study of invariant measures and the long time behavior of stochastic partial differential equations (SPDEs) driven by α -stable type noises has been extensively studied, we refer to [5, 6, 8, 12, 19] and the literatures therein.

In this paper, we firstly study the ergodicity of stochastic reaction-diffusion equation driven by subordinate Brownian motions, showing that the system (1.1) admits a unique invariant probability

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measure π . To do this, we need to prove the system is strong Feller and irreducible. Those two properties imply the uniqueness of the invariant measure according to Doob's theory (see [10]). To establish the strong Feller property, we truncate the nonlinearity and apply a gradient established in [7] or [30]. To establish the irreducibility, we need to prove the irreducibility of the stochastic evolution and then apply a control problem result in [26]. Unlike the case of SPDEs driven by cylindrical α -stable noises, the components of the noise are not independent, the approach in the proof of the irreducibility is very different from that in our previous paper [26].

Another topic is the large deviation principle (LDP) about the occupation measure. Let \mathcal{L}_t be the occupation measure of the system (1.1) given by

$$\mathcal{L}_t(A) := \frac{1}{t} \int_0^t \delta_{X_s}(A) ds \quad \text{for any measurable set } A, \quad (1.2)$$

where δ_a is the Dirac measure at a . By the uniqueness of invariant measure (see [2]), we know that the occupation measure \mathcal{L}_t converges to the invariant measure π . In this paper, we also study the LDP for the occupation measure \mathcal{L}_t . The LDP for empirical measures is one of the strongest ergodicity results for the long time behavior of Markov processes. It has been one of the classical research topics in probability since the pioneering work of Donsker and Varadhan [9]. Refer to the books [3, 4]. Based on the hyper-exponential recurrence criterion developed by Wu [28], we prove that the occupation measure \mathcal{L}_t obeys an LDP under τ -topology. As a consequence, we can obtain the exact rate of exponential ergodicity.

For stochastic partial differential equations, the problems of LDP have been extensively studied in recent years. Most of them, however, are concentrated on the small noise LDP of Freidlin-Wentzell type, which provide estimates for the probability that stochastic systems converge to their deterministic part as noises tend to zero. But there are only very few papers on the LDP of Donsker-Varadhan type for large time, which estimate the probability of the occupation measures' deviation from invariant measure. Gourcy [13, 14] established the LDP for occupation measures of stochastic Burgers and Navier-Stokes equations by the means of the hyper-exponential recurrence. Jakšić et al. [16] established the LDP for occupation measures of SPDE with smooth random perturbations by Kifer's LDP criterion [18]. Jakšić et al. [17] also gave the large deviations estimates for dissipative PDEs with rough noise by the hyper-exponential recurrence criterion. In [27], using the hyper-exponential recurrence criterion, an LDP for the occupation measure is derived for a class of non-linear monotone stochastic partial differential equations, such as stochastic p -Laplace equation, stochastic porous medium equation and stochastic fast-diffusion equation.

The paper is organized as follows. In Section 2, we give a brief review of some known results about the stochastic reaction-diffusion equations, and present the main result of this paper. In Sections 3 and 4, we prove the strong Feller property and the irreducibility of the system separately. In Section 5, we first recall the hyper-exponential criterion about the LDP for Markov processes, and then verify this condition by establishing some uniform estimates which also imply the tightness of the solution.

Throughout this paper, C_p is a positive constant depending on some parameter p , and C is a constant depending on no specific parameter (except α, β), whose value may be different from line to line by convention.

2 The model and the results

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be equipped with the usual Riemannian metric, and let $d\xi$ denote the Lebesgue measure on \mathbb{T} . For any $p \geq 1$, let

$$L^p(\mathbb{T}; \mathbb{R}) := \left\{ x : \mathbb{T} \rightarrow \mathbb{R}; \|x\|_{L^p} := \left(\int_{\mathbb{T}} |x(\xi)|^p d\xi \right)^{\frac{1}{p}} < \infty \right\}.$$

Denote

$$\mathbb{H} := \left\{ x \in L^2(\mathbb{T}; \mathbb{R}); \int_{\mathbb{T}} x(\xi) d\xi = 0 \right\}.$$

\mathbb{H} is a real separable Hilbert space with inner product

$$\langle x, y \rangle_{\mathbb{H}} := \int_{\mathbb{T}} x(\xi) y(\xi) d\xi, \quad \forall x, y \in \mathbb{H}.$$

Write $\|x\|_{\mathbb{H}} := (\langle x, x \rangle_{\mathbb{H}})^{\frac{1}{2}}$.

Let Δ be the Laplace operator on \mathbb{H} . Then $A := -\Delta$ is a positive self-adjoint operator on \mathbb{H} with the discrete spectral. More precisely, there exist an orthogonal basis $\{e_k; e_k = e^{i2\pi k\xi}, k \in \mathbb{Z}_*\}$ with $\mathbb{Z}_* := \mathbb{Z} \setminus \{0\}$, and a sequence of real numbers $\{\lambda_k = 4\pi^2|k|^2; k \in \mathbb{Z}_*\}$ such that $Ae_k = \lambda_k e_k$.

For any $\theta \geq 0$, let \mathbb{H}_θ be the domain of the fractional operator $A^{\frac{\theta}{2}}$, i.e.,

$$\mathbb{H}_\theta := \left\{ \sum_{k \in \mathbb{Z}_*} \lambda_k^{-\frac{\theta}{2}} a_k \cdot e_k : (a_k)_{k \in \mathbb{Z}_*} \subset \mathbb{R}, \sum_{k \in \mathbb{Z}_*} a_k^2 < +\infty \right\},$$

with the inner product

$$\langle u, v \rangle_\theta := \langle A^{\frac{\theta}{2}} u, A^{\frac{\theta}{2}} v \rangle_{\mathbb{H}} = \sum_{k \in \mathbb{Z}_*} \lambda_k^\theta \langle u, e_k \rangle_{\mathbb{H}} \cdot \langle v, e_k \rangle_{\mathbb{H}},$$

and with the norm $\|u\|_\theta := \langle u, u \rangle_\theta^{\frac{1}{2}} = \|A^{\frac{\theta}{2}} u\|_{\mathbb{H}}$. Clearly, \mathbb{H}_θ is densely and compactly embedded in \mathbb{H} . Particularly, let

$$\mathbb{V} := \mathbb{H}_1 \quad \text{and} \quad \|x\|_{\mathbb{V}} := \|x\|_1.$$

Let $\{W_t^k, t \geq 0\}_{k \in \mathbb{Z}_*}$ be a sequence of independent standard one-dimensional Brownian motion on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. The cylindrical Brownian motion on \mathbb{H} is defined by

$$W_t := \sum_{k \in \mathbb{Z}_*} W_t^k \cdot e_k.$$

For $\alpha \in (0, 2)$, let S_t be an independent $\alpha/2$ -stable subordinator, i.e., an increasing one dimensional Lévy process with Laplace transform

$$\mathbb{E} [e^{-\eta S_t}] = e^{-t|\eta|^{\alpha/2}}, \quad \eta > 0.$$

Then $L_t := W_{S_t}$ defines a subordinated cylindrical Brownian motion on \mathbb{H} . Refer to [1, 24].

For a sequence of bounded real numbers $\beta = (\beta_k)_{k \in \mathbb{N}}$, let us define

$$Q_\beta : \mathbb{H} \rightarrow \mathbb{H} \quad \text{such that} \quad Q_\beta u := \sum_{k \in \mathbb{Z}_*} \beta_k \langle u, e_k \rangle_{\mathbb{H}} \cdot e_k \quad \text{for } u \in \mathbb{H}.$$

We shall rewrite the system (1.1) into the following abstract form:

$$\begin{cases} dX_t + AX_t dt = N(X_t) dt + Q_\beta dL_t, \\ X_0 = x, \end{cases} \quad (2.1)$$

where

(i) the nonlinear term N is defined by

$$N(u) = u - u^3, \quad u \in \mathbb{H};$$

- (ii) $\{L_t\}_{t \geq 0}$ is a subordinated cylindrical Brownian motion on \mathbb{H} with $\alpha \in (1, 2)$, and the intensity Q_β satisfies that for some $\delta > 0$ and $\frac{3}{2} < \theta' \leq \theta < 2$,

$$\delta \lambda_k^{-\frac{\theta}{2}} \leq |\beta_k| \leq \delta^{-1} \lambda_k^{-\frac{\theta'}{2}}, \quad \forall k \in \mathbb{Z}_*.$$

Definition 2.1 We say that a predictable \mathbb{H} -valued stochastic process $X = (X_t^x)$ is a mild solution to Eq. (2.1), if for any $t \geq 0, x \in \mathbb{H}$, it holds (\mathbb{P} -a.s.):

$$X_t^x(\omega) = e^{-At}x + \int_0^t e^{-A(t-s)}N(X_s^x(\omega))ds + \int_0^t e^{-A(t-s)}Q_\beta dL_s(\omega). \quad (2.2)$$

By Lemma 3.1 in the next section, using the similar approach as in the proof of [29, Theorem 2.2], we can easily obtain that Eq. (2.1) admits a unique mild solution $X(\omega) \in \mathbb{D}([0, \infty); \mathbb{H}) \cap \mathbb{D}((0, \infty); \mathbb{V})$. Moreover, X is a Markov process.

Our first main result is the following theorem about the ergodicity of solution.

Theorem 2.2 Assume that $\alpha \in (1, 2)$. Then the Markov process X is strong Feller and irreducible in \mathbb{H} for any $t > 0$, and X admits a unique invariant measure.

Proof: We shall prove the the strong Feller property and irreducibility in Section 3 and Section 4. By the well-known Doob's Theorem (see [2]), we know that X admits at most one unique invariant probability measure. According to the Krylov-Bogolyubov's theorem (See [2]), if the family of the law $\{X_t; t \geq 1\}$ is tight, then there exists an invariant probability measure for (2.1). The tightness for $\{X_t; t \geq 1\}$ follows from Theorem 5.4.

The proof is complete. ■

Recall that \mathcal{L}_t defined by

$$\mathcal{L}_t(A) := \frac{1}{t} \int_0^t \delta_{X_s}(A)ds \quad \text{for any measurable set } A, \quad (2.3)$$

where δ_a is the Dirac measure at $a \in \mathbb{H}$. Then \mathcal{L}_t is in $\mathcal{M}_1(\mathbb{H})$, the space of probability measures on \mathbb{H} . On $\mathcal{M}_1(\mathbb{H})$, let $\sigma(\mathcal{M}_1(\mathbb{H}), \mathcal{B}_b(\mathbb{H}))$ be the τ -topology of convergence against measurable and bounded functions which is much stronger than the usual weak convergence topology $\sigma(\mathcal{M}_1(\mathbb{H}), C_b(\mathbb{H}))$, where $C_b(\mathbb{H})$ is the space of all bounded continuous functions on \mathbb{H} . See [9] or [3, Section 6.2].

Our second main result is about the LDP for occupation time \mathcal{L}_t , whose proof will be given in the last section.

Theorem 2.3 Assume that $\alpha \in (1, 2)$. Then the family $\mathbb{P}_\nu(\mathcal{L}_T \in \cdot)$ as $T \rightarrow +\infty$ satisfies the LDP with respect to the τ -topology, with speed T and rate function J defined by (5.1) below, uniformly for any initial measure ν in $\mathcal{M}_1(\mathbb{H})$. More precisely, the following three properties hold:

- (a1) for any $a \geq 0$, $\{\mu \in \mathcal{M}_1(\mathbb{H}); J(\mu) \leq a\}$ is compact in $(\mathcal{M}_1(\mathbb{H}), \tau)$;
- (a2) (the lower bound) for any open set G in $(\mathcal{M}_1(\mathbb{H}), \tau)$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \log \inf_{\nu \in \mathcal{M}_1(\mathbb{H})} \mathbb{P}_\nu(\mathcal{L}_T \in G) \geq -\inf_G J;$$

- (a3) (the upper bound) for any closed set F in $(\mathcal{M}_1(\mathbb{H}), \tau)$,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log \sup_{\nu \in \mathcal{M}_1(\mathbb{H})} \mathbb{P}_\nu(\mathcal{L}_T \in F) \leq -\inf_F J.$$

Remark 2.4 For every $f : \mathbb{H} \rightarrow \mathbb{R}$ measurable and bounded, as $\nu \rightarrow \int_{\mathbb{H}} f d\nu$ is continuous w.r.t. the τ -topology, then by the contraction principle ([3, Theorem 4.2.1]),

$$\mathbb{P}_\nu \left(\frac{1}{T} \int_0^T f(X_s) ds \in \cdot \right)$$

satisfies the LDP on \mathbb{R} uniformly for any initial measure ν in $\mathcal{M}_1(\mathbb{H})$, with the rate function given by

$$J^f(r) = \inf \left\{ J(\mu) < +\infty \mid \mu \in \mathcal{M}_1(\mathbb{H}) \text{ and } \int f d\mu = r \right\}, \quad \forall r \in \mathbb{R}.$$

3 Strong Feller property

3.1 Some useful estimates

We shall often use the following inequalities (see [29]):

$$\|A^{\sigma_1} x\|_{\mathbb{H}} \leq C_{\sigma_1, \sigma_2} \|A^{\sigma_2} x\|_{\mathbb{H}}, \quad \forall \sigma_1 \leq \sigma_2, x \in \mathbb{H}; \quad (3.1)$$

$$\|A^\sigma e^{-At}\|_{\mathbb{H}} \leq C_\sigma t^{-\sigma}, \quad \forall \sigma > 0, t > 0; \quad (3.2)$$

$$\|x\|_{L^4}^4 \leq \|x\|_{\mathbb{V}}^2 \|x\|_{\mathbb{H}}^2, \quad \forall x \in \mathbb{V}; \quad (3.3)$$

$$\langle x, N(x) \rangle_{\mathbb{H}} \leq \frac{1}{4}, \quad \forall x \in \mathbb{H}; \quad (3.4)$$

$$\|N(x)\|_{\mathbb{V}} \leq C(\|x\|_{\mathbb{V}} + \|x\|_{\mathbb{V}}^3), \quad \forall x \in \mathbb{V}; \quad (3.5)$$

$$\|N(x) - N(y)\|_{\mathbb{V}} \leq C(1 + \|x\|_{\mathbb{V}}^2 + \|y\|_{\mathbb{V}}^2) \cdot \|x - y\|_{\mathbb{V}}, \quad \forall x, y \in \mathbb{V}; \quad (3.6)$$

$$\|N(x) - N(y)\|_{\mathbb{H}} \leq C(1 + \|A^{\frac{1}{4}} x\|_{\mathbb{H}}^2 + \|A^{\frac{1}{4}} y\|_{\mathbb{H}}^2) \cdot \|x - y\|_{\mathbb{H}}, \quad \forall x, y \in \mathbb{H}. \quad (3.7)$$

For all $\sigma \geq \frac{1}{6}$,

$$\|N(x) - N(y)\|_{\mathbb{H}} \leq C(1 + \|A^\sigma x\|_{\mathbb{H}}^2 + \|A^\sigma y\|_{\mathbb{H}}^2) \cdot \|A^\sigma(x - y)\|_{\mathbb{H}}, \quad \forall x, y \in \mathbb{H}; \quad (3.8)$$

$$\|N(x)\|_{\mathbb{H}} \leq C(1 + \|A^\sigma x\|_{\mathbb{H}}^3), \quad \forall x \in \mathbb{H}. \quad (3.9)$$

Let us now consider the following stochastic convolution:

$$Z_t := \int_0^t e^{-(t-s)A} Q_\beta dL_s = \sum_{k \in \mathbb{Z}_*} \int_0^t e^{-(t-s)\lambda_k} \beta_k dW_{S_s}^k \cdot e_k. \quad (3.10)$$

The estimate about Z_t will play an important role in next sections (cf. [20, 22]).

Lemma 3.1 [7, Lemma 2.3] Suppose that for some $\gamma \in \mathbb{R}$,

$$K_\gamma := \sum_{k \in \mathbb{Z}_*} \lambda_k^\gamma |\beta_k|^2 < +\infty.$$

Then for any $p \in (0, \alpha)$ and $T > 0$,

$$\sup_{t \in [0, T]} \mathbb{E} [\|Z_t\|_{\gamma+1}^p] \leq C_{\alpha, p} K_\gamma^{\frac{p}{2}} T^{\frac{p}{\alpha} - \frac{p}{2}}; \quad (3.11)$$

for any $\theta < \gamma$,

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|Z_t\|_\theta^p \right] \leq C_{\alpha, p} K_\gamma^{\frac{p}{2}} T^{\frac{p}{\alpha}} \left(1 + T^{\frac{\gamma - \theta}{2}} \right); \quad (3.12)$$

for any $\varepsilon > 0$,

$$\mathbb{P} \left(\sup_{t \in [0, T]} \|Z_t\|_\theta \leq \varepsilon \right) > 0. \quad (3.13)$$

Moreover, $t \mapsto Z_t$ is almost surely càdlàg in \mathbb{H}_θ .

3.2 Strong Feller property

For any $f \in \mathcal{B}_b(\mathbb{H})$, $t \geq 0$ and $x \in \mathbb{H}$, define

$$P_t f(x) := \mathbb{E}[f(X_t^x)].$$

The main result of this section is

Theorem 3.2 $(P_t)_{t \geq 0}$, as a semigroup on $\mathcal{B}_b(\mathbb{H})$, is strong Feller.

To prove Theorem 3.2, thanks to a standard argument (see [29, p. 943] for example), we only need to prove that the following lemma.

Lemma 3.3 $(P_t)_{t \geq 0}$, as a semigroup on $\mathcal{B}_b(\mathbb{V})$, is strong Feller.

Proof: The proof is inspired by the proof of Theorem [29, Theorem 6.2]. Let $T_0 > 0$ be arbitrary, it suffices to show that for all $t \in (0, T_0]$, $x \in \mathbb{V}$ and $f \in \mathcal{B}_b(\mathbb{V})$,

$$\lim_{\|y - x\|_{\mathbb{V}} \rightarrow 0} P_t f(y) = P_t f(x). \quad (3.14)$$

Without loss of generality, we assume $\|f\|_\infty := \sup_{x \in \mathbb{V}} |f(x)| = 1$. We divide the proof into three steps.

Step 1. Since the nonlinearity N is not bounded and Lipschitz continuous, we need to use a truncation technique. Consider the equation with truncated nonlinearity as follows:

$$dX_t^\rho + AX_t^\rho dt = N^\rho(X_t^\rho) dt + Q_\beta dL_t, \quad X_0^\rho = x \in \mathbb{V}, \quad (3.15)$$

where $\rho > 0$, $N^\rho(x) := N(x)\chi(\|x\|_{\mathbb{V}}/\rho)$ for all $x \in \mathbb{V}$ and $\chi : \mathbb{R} \rightarrow [0, 1]$ is a smooth function such that

$$\chi(z) = 1 \quad \text{for } |z| \leq 1, \quad \chi(z) = 0 \quad \text{for } |z| \geq 2.$$

By (3.5), for all $x \in \mathbb{V}$,

$$\|N^\rho(x)\|_{\mathbb{V}} \leq C(\|x\|_{\mathbb{V}}^3 + \|x\|_{\mathbb{V}})\chi(\|x\|_{\mathbb{V}}/\rho) \leq C(\rho^3 + \rho). \quad (3.16)$$

It follows from (3.6) that

$$\|N^\rho(x) - N^\rho(y)\|_{\mathbb{V}} \leq C(1 + \rho^2) \cdot \|x - y\|_{\mathbb{V}}. \quad (3.17)$$

Hence, Eq. (3.15) admits a unique Markov solution $X_t^\rho \in \mathbb{D}([0, \infty); \mathbb{V})$.

By Theorem 3.1 in [7] (choosing $\sigma = \gamma = 1$ and $\gamma' = 0$ there), we have for any $0 < t \leq T_0$, and $x, y \in \mathbb{V}$,

$$|\mathbb{E}[f(X_t^{\rho, x})] - \mathbb{E}[f(X_t^{\rho, y})]| \leq Ct^{-\frac{1}{\alpha} - \frac{\theta - 1}{2}} \|f\|_\infty \cdot \|x - y\|_{\mathbb{V}}. \quad (3.18)$$

Step 2. Define

$$K_{T_0}(\omega) := \sup_{0 \leq t \leq T_0} \|Z_t(\omega)\|_{\mathbb{V}}, \quad \omega \in \Omega.$$

By Lemma 3.1 and Markov inequality, we have

$$\mathbb{P}(K_{T_0} > \rho/2) \leq C(\alpha, T_0)/\rho, \quad (3.19)$$

where $C(\alpha, T_0)$ is some constant depending on α and T_0 .

Choose ρ so large that $\|x\|_{\mathbb{V}} \leq \sqrt{\rho} < \rho/2 - 1$ and define

$$G := \{K_{T_0} \leq \rho/2\}.$$

For all $\omega \in \Omega$, define $Y_t(\omega) := X_t(\omega) - Z_t(\omega)$, then

$$dY_t + AY_t dt = N(Y_t + Z_t)dt, \quad Y_0 = x \in \mathbb{V}.$$

By (ii) of Lemma 4.1 in [29], there exists some $0 < t_0 \leq T_0$ depending on ρ such that for all $\omega \in G$,

$$\sup_{0 \leq t \leq t_0} \|Y_t^x(\omega)\|_{\mathbb{V}} \leq 1 + \|x\|_{\mathbb{V}} \leq 1 + \sqrt{\rho} < \rho/2.$$

Then

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \leq t \leq t_0} \|X_t^x\|_{\mathbb{V}} \geq \rho\right) &\leq \mathbb{P}\left(\sup_{0 \leq t \leq t_0} \|Y_t^x\|_{\mathbb{V}} + \sup_{0 \leq t \leq T_0} \|Z_t\|_{\mathbb{V}} \geq \rho\right) \\ &\leq \mathbb{P}(K_{T_0} > \rho/2) + \mathbb{P}\left(\sup_{0 \leq t \leq t_0} \|Y_t^x\|_{\mathbb{V}} > \rho/2, G\right) \\ &= \mathbb{P}(K_{T_0} > \rho/2). \end{aligned} \quad (3.20)$$

The above inequality, together with (3.19) and (3.20), implies that

$$\mathbb{P}\left(\sup_{0 \leq t \leq t_0} \|X_t^x\|_{\mathbb{V}} \geq \rho\right) \leq \mathbb{P}(K_{T_0} > \rho/2) \leq C(\alpha, T_0)/\rho. \quad (3.21)$$

Step 3. Define the stopping time

$$\tau_x := \inf\{t > 0; \|X_t^x\|_{\mathbb{V}} \geq \rho\}.$$

By (3.21), we obtain that for all $t \in [0, t_0]$,

$$\mathbb{P}_x(\tau_x \leq t) = \mathbb{P}\left(\sup_{0 \leq s \leq t} \|X_s^x\|_{\mathbb{V}} \geq \rho\right) \leq C(\alpha, t)/\rho. \quad (3.22)$$

Since Eqs. (2.1) and (3.15) both have a unique mild solution, for all $t \in [0, \tau_x]$, we have

$$X_t^{\rho, x} = X_t^x \quad a.s.. \quad (3.23)$$

Let $y \in \mathbb{V}$ be such that $\|x - y\|_{\mathbb{V}} \leq 1$ and choose $\rho > 0$ be sufficiently large so that $\max\{\|x\|_{\mathbb{V}}, \|y\|_{\mathbb{V}}\} \leq \sqrt{\rho}$. For any $t \in (0, t_0]$, it holds that

$$|P_t f(x) - P_t f(y)| = |\mathbb{E}[f(X_t^x)] - \mathbb{E}[f(X_t^y)]| = I_1 + I_2 + I_3, \quad (3.24)$$

where

$$\begin{aligned} I_1 &:= |\mathbb{E}[f(X_t^x)1_{[\tau_x > t]}] - \mathbb{E}[f(X_t^y)1_{[\tau_y > t]}]|, \\ I_2 &:= |\mathbb{E}[f(X_t^x)1_{[\tau_x \leq t]}]|, \end{aligned}$$

$$I_3 := |\mathbb{E}[f(X_t^y)1_{[\tau_y \leq t]}]|.$$

It follows from (3.22) that

$$I_2 \leq \frac{C\|f\|_\infty}{\rho}, \quad I_3 \leq \frac{C\|f\|_\infty}{\rho}. \quad (3.25)$$

It remains to estimate I_1 . It follows from (3.18), (3.22) and (3.23) that

$$\begin{aligned} I_1 &= |\mathbb{E}[f(X_t^{\rho,x})1_{[\tau_x > t]}] - \mathbb{E}[f(X_t^{\rho,y})1_{[\tau_y > t]}]| \\ &\leq |\mathbb{E}[f(X_t^{\rho,x})] - \mathbb{E}[f(X_t^{\rho,y})]| + |\mathbb{E}[f(X_t^{\rho,x})1_{[\tau_x \leq t]}]| + |\mathbb{E}[f(X_t^{\rho,y})1_{[\tau_y \leq t]}]| \\ &\leq Ct^{-\frac{1}{\alpha} - \frac{\theta-1}{2}} \|f\|_\infty \cdot \|x - y\|_{\mathbb{V}} + 2C\|f\|_\infty/\rho. \end{aligned} \quad (3.26)$$

For all $\varepsilon > 0, t \in (0, t_0]$, choosing

$$\rho \geq \max \left\{ \frac{12C\|f\|_\infty}{\varepsilon}, 2\|x\|_{\mathbb{V}}^2 + 2 \right\}, \quad \delta = \frac{\varepsilon}{2C} t^{\frac{1}{\alpha} + \frac{\theta-1}{2}},$$

by Eqs. (3.24), (3.25) and (3.26), we obtain that for all $\|x - y\|_{\mathbb{V}} \leq \delta$,

$$|P_t f(x) - P_t f(y)| < \varepsilon.$$

As $t_0 < t \leq T_0$, it follows from the Markov property and the strong Feller property above that

$$P_t f(y) - P_t f(x) = P_{t_0}[P_{t-t_0} f](y) - P_{t_0}[P_{t-t_0} f](x) \rightarrow 0,$$

as $\|y - x\|_{\mathbb{V}} \rightarrow 0$.

The proof is complete. ■

4 Irreducibility

The main result of this part is the irreducibility of the stochastic dynamics.

Theorem 4.1 *Assume that $\alpha \in (1, 2)$. For any initial value $x \in \mathbb{H}$, the Markov process $X = \{X_t^x\}_{t \geq 0, x \in \mathbb{H}}$ to Eq. (2.1) is irreducible in \mathbb{H} .*

Remark 4.2 *By the well-known Doob's Theorem (see [2]), the strong Feller property and the irreducibility imply that X admits at most one unique invariant probability measure.*

4.1 Irreducibility of stochastic convolution

Let \mathbb{S} be the space of all increasing and càdlàg functions from $(0, \infty)$ to $(0, \infty)$ with $\lim_{s \rightarrow 0+} l_s = 0$, which is endowed with the Skorohod metric and the probability measure $\mu_{\mathbb{S}}$ so that the coordinate process $S_t(l) := l_t$ is an $\alpha/2$ -stable subordinator.

Consider the following product probability space

$$(\Omega, \mathcal{F}, \mathbb{P}) := (\mathbb{W} \times \mathbb{S}, \mathcal{B}(\mathbb{W}) \times \mathcal{B}(\mathbb{S}), \mu_{\mathbb{W}} \times \mu_{\mathbb{S}})$$

and define

$$L_t(w, l) := w_{l_t}.$$

We shall use the following two natural filtration associated with the Lévy process L_t and the Brownian motion W_t :

$$\mathcal{F}_t := \sigma\{L_s(w, l); s \leq t\}, \quad \mathcal{F}_t^{\mathbb{W}} := \sigma\{W_s(w); s \leq t\},$$

and denote by $\mathbb{E}^{\mathbb{S}}$ and $\mathbb{E}^{\mathbb{W}}$ the partial integrations with respect to S and W , respectively.

For any $l \in \mathbb{S}$, let Z_t^l solve the following equation:

$$dZ_t^l + AZ_t^l dt = Q_\beta dW_{l_t}, \quad Z_0^l = 0. \quad (4.1)$$

It is well known that

$$Z_t^l = \int_0^t e^{-A(t-s)} Q_\beta dW_{l_s} = \sum_{k \in \mathbb{Z}_*} \beta_k z_k(t) \cdot e_k,$$

where

$$z_k(t) = \int_0^t e^{-\lambda_k(t-s)} dW_{l_s}^k.$$

Notice that for any fixed $l \in \mathbb{S}$, $\{z_k\}_{k \in \mathbb{Z}_*}$ are independent by the independence of $\{W^k\}_{k \in \mathbb{Z}_*}$.

We claim that for any $\gamma \in (0, \theta' - 1)$ with θ' defined above Definition 2.1,

$$\mathbb{E}^W \left[\sup_{0 \leq t \leq T} \|A^\gamma Z_t^l\|_{\mathbb{H}} \right] \leq C_{\gamma, \theta'} \sqrt{l_T}. \quad (4.2)$$

Indeed, upper to a standard finite dimension approximation argument, using integration by parts we get

$$Z_t^l = Q_\beta W_{l_t} - \int_0^t A e^{-A(t-s)} Q_\beta W_{l_s} ds, \quad (4.3)$$

which clearly implies

$$\begin{aligned} \sup_{0 \leq t \leq T} \|A^\gamma Z_t^l\|_{\mathbb{H}} &\leq \sup_{0 \leq t \leq T} \|A^\gamma Q_\beta W_{l_t}\|_{\mathbb{H}} + \sup_{0 \leq t \leq T} \int_0^t \|A^{1+\gamma} e^{-A(t-s)} Q_\beta W_{l_s}\|_{\mathbb{H}} ds \\ &\leq \sup_{0 \leq t \leq T} \|A^\gamma Q_\beta W_{l_t}\|_{\mathbb{H}} + \sup_{0 \leq t \leq T} \int_0^t \|A^{1+\gamma-\gamma'} e^{-A(t-s)}\| \cdot \|A^{\gamma'} Q_\beta W_{l_s}\|_{\mathbb{H}} ds \\ &\leq \sup_{0 \leq t \leq T} \|A^\gamma Q_\beta W_{l_t}\|_{\mathbb{H}} + C \sup_{0 \leq t \leq T} \|A^{\gamma'} Q_\beta W_{l_t}\|_{\mathbb{H}} \cdot \sup_{0 \leq t \leq T} \int_0^t (t-s)^{1+\gamma-\gamma'} ds \\ &\leq \sup_{0 \leq t \leq l_T} \|A^\gamma Q_\beta W_t\|_{\mathbb{H}} + CT^{\gamma'-\gamma} \sup_{0 \leq t \leq l_T} \|A^{\gamma'} Q_\beta W_t\|_{\mathbb{H}}, \end{aligned}$$

where $\gamma' \in (\gamma, \theta' - 1)$. Hence, by the martingale inequality we get (4.2).

The following lemma is concerned with the support of the distribution of $(\{Z_t\}_{0 \leq t \leq T}, Z_T)$.

Lemma 4.3 *For any $T > 0, 0 < p < \infty$, the random variable $(\{Z_t\}_{0 \leq t \leq T}, Z_T)$ has a full support in $L^p([0, T]; \mathbb{V}) \times \mathbb{V}$. More precisely, for any $\phi \in L^p([0, T]; \mathbb{V}), a \in \mathbb{V}, \varepsilon > 0$,*

$$\mathbb{P} \left(\int_0^T \|Z_t - \phi_t\|_{\mathbb{V}}^p dt + \|Z_T - a\|_{\mathbb{V}} < \varepsilon \right) > 0. \quad (4.4)$$

Proof: The proof is divided into several steps.

Step 1. (Finite dimensional projection) For any $N \in \mathbb{N}$, let \mathbb{H}_N be the Hilbert space spanned by $\{e_k\}_{1 \leq |k| \leq N}$, and let $\pi_N : \mathbb{H} \rightarrow \mathbb{H}_N$ be the orthogonal projection. Notice that π_N is also an orthogonal projection in \mathbb{V} . Define

$$\pi^N := I - \pi_N, \quad \mathbb{H}^N := \pi^N \mathbb{H}.$$

Then for any given $l \in \mathbb{S}$, $\pi_N Z^l$ and $\pi^N Z^l$ are independent. Thus, for any $\phi \in L^p([0, T]; \mathbb{V})$ and $a \in \mathbb{V}$, we have

$$\mathbb{P} \left(\int_0^T \|Z_t - \phi_t\|_{\mathbb{V}}^p dt + \|Z_T - a\|_{\mathbb{V}} < \varepsilon \right)$$

$$\begin{aligned}
&= \mathbb{E}^{\mathbb{S}} \left(\mathbb{P}^{\mathbb{W}} \left(\int_0^T \|Z_t^l - \phi_t\|_{\mathbb{V}}^p dt + \|Z_T^l - a\|_{\mathbb{V}} < \varepsilon \right) \Big|_{l=S} \right) \\
&\geq \mathbb{E}^{\mathbb{S}} \left(\mathbb{P}^{\mathbb{W}} \left(\int_0^T \|\pi_N(Z_t^l - \phi_t)\|_{\mathbb{V}}^p dt + \|\pi_N(Z_T^l - a)\|_{\mathbb{V}} < \frac{\varepsilon}{2^{p+1}}, \right. \right. \\
&\quad \left. \left. \int_0^T \|\pi^N(Z_t^l - \phi_t)\|_{\mathbb{V}}^p dt + \|\pi^N(Z_T^l - a)\|_{\mathbb{V}} < \frac{\varepsilon}{2^{p+1}} \right) \Big|_{l=S} \right) \\
&= \mathbb{E}^{\mathbb{S}} \left(\mathbb{P}^{\mathbb{W}} \left(\int_0^T \|\pi_N(Z_t^l - \phi_t)\|_{\mathbb{V}}^p dt + \|\pi_N(Z_T^l - a)\|_{\mathbb{V}} < \frac{\varepsilon}{2^{p+1}} \right) \Big|_{l=S} \right. \\
&\quad \left. \times \mathbb{P}^{\mathbb{W}} \left(\int_0^T \|\pi^N(Z_t^l - \phi_t)\|_{\mathbb{V}}^p dt + \|\pi^N(Z_T^l - a)\|_{\mathbb{V}} < \frac{\varepsilon}{2^{p+1}} \right) \Big|_{l=S} \right).
\end{aligned}$$

For any $\gamma \in (\frac{1}{2}, \theta' - 1)$, by the spectral gap inequality, Chebyshev inequality and (4.2), we have for any $\eta > 0$

$$\begin{aligned}
\mathbb{P}^{\mathbb{W}} \left(\sup_{0 \leq t \leq T} \|\pi^N Z_t^l\|_{\mathbb{V}} > \eta \right) &\leq \mathbb{P}^{\mathbb{W}} \left(\sup_{0 \leq t \leq T} \|\pi^N A^\gamma Z_t^l\|_{\mathbb{H}} > \eta \lambda_N^{\gamma - \frac{1}{2}} \right) \\
&\leq \mathbb{P}^{\mathbb{W}} \left(\sup_{0 \leq t \leq T} \|A^\gamma Z_t^l\|_{\mathbb{H}} > \eta \lambda_N^{\gamma - \frac{1}{2}} \right) \\
&\leq C_{\theta', \gamma} \sqrt{l_T} \eta^{-1} \lambda_N^{\frac{1}{2} - \gamma}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbb{P} \left(\int_0^T \|Z_t - \phi_t\|_{\mathbb{V}}^p dt + \|Z_T - a\|_{\mathbb{V}} < \varepsilon \right) \\
&\geq \mathbb{E}^{\mathbb{S}} \left[\mathbb{P}^{\mathbb{W}} \left(\int_0^T \|\pi_N(Z_t^l - \phi_t)\|_{\mathbb{V}}^p dt + \|\pi_N(Z_T^l - a)\|_{\mathbb{V}} < \frac{\varepsilon}{2^{p+1}} \right) \Big|_{l=S} \times \left(1 - C_{\theta', \gamma} \sqrt{l_T} 2^{p+1} \varepsilon^{-1} \lambda_N^{\frac{1}{2} - \gamma} \right) \Big|_{l=S} \right] \\
&\geq \mathbb{E}^{\mathbb{S}} \left[\mathbb{P}^{\mathbb{W}} \left(\int_0^T \|\pi_N(Z_t^l - \phi_t)\|_{\mathbb{V}}^p dt + \|\pi_N(Z_T^l - a)\|_{\mathbb{V}} < \frac{\varepsilon}{2^{p+1}} \right) \Big|_{l=S} \times \left(1 - C_{\theta', \gamma} T 2^{p+1} \varepsilon^{-1} \lambda_N^{\frac{1}{2} - \gamma} \right), S_T \leq T^2 \right] \\
&\geq \frac{1}{2} \mathbb{E}^{\mathbb{S}} \left[\mathbb{P}^{\mathbb{W}} \left(\int_0^T \|\pi_N(Z_t^l - \phi_t)\|_{\mathbb{V}}^p dt + \|\pi_N(Z_T^l - a)\|_{\mathbb{V}} < \frac{\varepsilon}{2^{p+1}} \right) \Big|_{l=S}, S_T \leq T^2 \right],
\end{aligned}$$

as N is sufficiently large.

As long as we prove that for any $\varepsilon > 0$

$$\mathbb{E}^{\mathbb{S}} \left[\mathbb{P}^{\mathbb{W}} \left(\int_0^T \|\pi_N(Z_t^l - \phi_t)\|_{\mathbb{V}}^p dt + \|\pi_N(Z_T^l - a)\|_{\mathbb{V}} < \varepsilon \right) \Big|_{l=S}, S_T \leq T^2 \right] > 0, \quad (4.5)$$

the proof is complete.

Step 2. It remains to prove (4.5). Since $\pi_N Z_t^l = \sum_{|i| \leq N} z_i(t) e_i$ with $\{z_i(t)\}_i$ being independent stochastic processes, it suffices to prove (4.5) for one dimensional case, i.e., for any $\phi \in L^p([0, T]; \mathbb{R})$, $a \in \mathbb{R}$ and $\varepsilon > 0$,

$$\mathbb{E}^{\mathbb{S}} \left[\mathbb{P}^{\mathbb{W}} \left(\int_0^T |z(t) - \phi(t)|^p dt + |z(T) - a| < \varepsilon \right) \Big|_{l=S}, S_T \leq T^2 \right] > 0, \quad (4.6)$$

where $z(t) = \int_0^t e^{-\lambda(t-s)} dw_{l_s}$ with $\lambda > 0$ and w_t being a one dimensional Brownian motion. To prove (4.6), we only need to show that

$$\mathbb{E}^{\mathbb{S}} \left[\mathbb{P}^{\mathbb{W}} \left(\int_0^T \left| \int_0^t e^{\lambda s} dw_{l_s} - e^{\lambda t} \phi(t) \right|^p dt + \left| \int_0^T e^{\lambda s} dw_{l_s} - e^{\lambda T} a \right| < \varepsilon \right) \Big|_{l=S}, S_T \leq T^2 \right] > 0, \quad (4.7)$$

Since the simple function space is dense in $L^p([0, T]; \mathbb{R})$, without loss of generality, we assume that $e^{\lambda t} \phi(t)$ is a simple function vanished at $t = 0$ and having the form:

$$e^{\lambda t} \phi(t) = \sum_{j=0}^{m-1} a_j 1_{[t_j, t_{j+1})}(t) + a_m 1_{\{t_m\}}(t) \quad (4.8)$$

where $0 = t_0 < t_1 < \dots < t_m = T$ and $a_0 = 0, a_1 \in \mathbb{R}, \dots, a_{m-1} \in \mathbb{R}$ and $a_m = e^{\lambda T} a$.

Define $M := \sup_{1 \leq k \leq m} |a_k|$ and

$$\Delta_{t_0, \dots, t_m, \sigma, \delta} = \{l \in \mathbb{S} : 0 = \tau_0 < \tau_1 < \tau_2 < \dots < \tau_{m-1} < \tau_m < T \text{ such that } l_T \leq T^2 \\ \tau_i \in (t_i - \sigma, t_i + \sigma), l_{\tau_i} - l_{\tau_i-} \in (\delta, 2\delta) \text{ for } 0 \leq i \leq m\}.$$

It is easy to see for all $\sigma > 0, \delta > 0$,

$$\mathbb{P}^{\mathbb{S}}(\Delta_{t_0, \dots, t_m, \sigma, \delta}) > 0. \quad (4.9)$$

Choosing $\sigma < \frac{1}{2} \min_{1 \leq k \leq m} (t_k - t_{k-1}) \leq \frac{T}{2}$, we immediately get

$$t_k - \sigma < \tau_k < t_k + \sigma < t_{k+1} - \sigma < \tau_{k+1} < t_{k+1} + \sigma, \quad k = 0, 1, \dots, m-1. \quad (4.10)$$

Denote

$$\Delta a_j := a_j - a_{j-1}, \quad \Delta w_j := w_{l_{\tau_j}} - w_{l_{\tau_j-}}, \quad I_j := \sup_{\tau_j \leq t < \tau_{j+1}} \left| \int_{\tau_{j-1}}^{\tau_j-} e^{\lambda s} dw_{l_s} \right|^p.$$

Notice that $e^{\lambda T} a = a_m = \sum_{j=1}^m \Delta a_j$, it is easy to check

$$\left| \int_0^T e^{\lambda s} dw_{l_s} - e^{\lambda T} a \right| = \left| \sum_{j=1}^m \left(\int_{\tau_{j-1}}^{\tau_j-} e^{\lambda s} dw_{l_s} + e^{\lambda \tau_j} \Delta w_j \right) + \int_{\tau_m}^T e^{\lambda s} dw_{l_s} - a_m \right| \\ \leq \sum_{j=1}^m |e^{\lambda \tau_j} \Delta w_j - \Delta a_j| + \sum_{j=1}^m I_j^{1/p} + \left| \int_{\tau_m}^T e^{\lambda s} dw_{l_s} \right|, \quad (4.11)$$

and

$$\int_0^T \left| \int_0^t e^{\lambda s} dw_{l_s} - e^{\lambda t} \phi(t) \right|^p dt \\ = \sum_{k=1}^m \int_{\tau_{k-1}}^{\tau_k} \left| \int_0^t e^{\lambda s} dw_{l_s} - e^{\lambda t} \phi(t) \right|^p dt + \int_{\tau_m}^T |e^{\lambda t} z(t) - e^{\lambda t} \phi(t)|^p dt. \quad (4.12)$$

Since $\phi(t) = 0$ for all $t \in [0, t_1)$, we have

$$\int_0^{\tau_1} \left| \int_0^t e^{\lambda s} dw_{l_s} - e^{\lambda t} \phi(t) \right|^p dt = \int_0^{\tau_1} \left| \int_0^t e^{\lambda s} dw_{l_s} \right|^p dt \leq 2^{p-1} T (I_1 + e^{\lambda p \tau_1} |\Delta w_1|^p). \quad (4.13)$$

For $t \in [\tau_k, \tau_{k+1})$ with $k = 1, \dots, m-1$, we have

$$\int_0^t e^{\lambda s} dw_{l_s} = \sum_{j=1}^k \left[\int_{\tau_{j-1}}^{\tau_j-} e^{\lambda s} dw_{l_s} + e^{\lambda \tau_j} \Delta w_j \right] + \int_{\tau_k}^t e^{\lambda s} dw_{l_s}. \quad (4.14)$$

Let us now compare $[\tau_k, \tau_{k+1})$ with $[t_k, t_{k+1})$, it is easy to see that

$$\int_{\tau_k}^{\tau_{k+1}} \left| \int_0^t e^{\lambda s} dw_{l_s} - e^{\lambda t} \phi(t) \right|^p dt \leq 3^{p-1} (J_{k1} + J_{k2} + J_{k3}), \quad (4.15)$$

where

$$\begin{aligned} J_{k1} &:= \left(\int_{\tau_k}^{t_k} \left| \int_0^t e^{\lambda s} dw_{l_s} - e^{\lambda t} \phi(t) \right|^p dt \right) 1_{\{\tau_k < t_k\}}, \\ J_{k2} &:= \int_{t_k}^{t_{k+1} \wedge \tau_{k+1}} \left| \int_0^t e^{\lambda s} dw_{l_s} - e^{\lambda t} \phi(t) \right|^p dt, \\ J_{k3} &:= \left(\int_{t_{k+1}}^{\tau_{k+1}} \left| \int_0^t e^{\lambda s} dw_{l_s} - e^{\lambda t} \phi(t) \right|^p dt \right) 1_{\{t_{k+1} < \tau_{k+1}\}}. \end{aligned}$$

By the easy relation $a_{k-1} = \sum_{j=1}^k \Delta a_j - \Delta a_k$, (4.10) and (4.14), we further obtain

$$\begin{aligned} J_{k1} &= \left(\int_{\tau_k}^{t_k} \left| \sum_{j=1}^k \int_{\tau_{j-1}}^{\tau_j} e^{\lambda s} dw_{l_s} + \sum_{j=1}^k (e^{\lambda \tau_j} \Delta w_j - \Delta a_j) + \int_{\tau_k}^t e^{\lambda s} dw_{l_s} + \Delta a_k \right|^p dt \right) 1_{\{\tau_k < t_k\}} \\ &\leq \sigma(2k+2)^{p-1} \left[\sum_{j=1}^k I_j + \sum_{j=1}^k |e^{\lambda \tau_j} \Delta w_j - \Delta a_j|^p + \sup_{\tau_k \leq t < \tau_{k+1}} \left| \int_{\tau_k}^t e^{\lambda s} dw_{l_s} \right|^p + |\Delta a_k|^p \right] \\ &= \sigma(2k+2)^{p-1} \left[\sum_{j=1}^{k+1} I_j + \sum_{j=1}^k |e^{\lambda \tau_j} \Delta w_j - \Delta a_j|^p + |\Delta a_k|^p \right]. \end{aligned}$$

For J_{k2} , by the similar argument, we have

$$\begin{aligned} J_{k2} &\leq (2k+1)^{p-1} (t_{k+1} - t_k) \left[\sum_{j=1}^k I_j + \sum_{j=1}^k |e^{\lambda \tau_j} \Delta w_j - \Delta a_j|^p + \sup_{\tau_k \leq t < \tau_{k+1}} \left| \int_{\tau_k}^t e^{\lambda s} dw_{l_s} \right|^p \right] \\ &= (2k+1)^{p-1} (t_{k+1} - t_k) \left[\sum_{j=1}^{k+1} I_j + \sum_{j=1}^k |e^{\lambda \tau_j} \Delta w_j - \Delta a_j|^p \right]. \end{aligned}$$

Moreover, as $t_{k+1} < \tau_{k+1}$ we similarly have

$$J_{k3} \leq \sigma(2k+2)^{p-1} \left[\sum_{j=1}^{k+1} I_j + \sum_{j=1}^k |e^{\lambda \tau_j} \Delta w_j - \Delta a_j|^p + |\Delta a_{k+1}|^p \right]. \quad (4.16)$$

Therefore, for $k = 1, \dots, m-1$,

$$\begin{aligned} &\int_{\tau_k}^{\tau_{k+1}} \left| \int_0^t e^{\lambda s} dw_{l_s} - e^{\lambda t} \phi(t) \right|^p dt \\ &\leq 2T(2k+2)^{p-1} \left[\sum_{j=1}^{k+1} I_j + \sum_{j=1}^k |e^{\lambda \tau_j} \Delta w_j - \Delta a_j|^p \right] + 2\sigma(2k+2)^{p-1} (|\Delta a_k|^p + |\Delta a_{k+1}|^p). \end{aligned} \quad (4.17)$$

Similarly, we have

$$\begin{aligned} &\int_{\tau_m}^T \left| \int_0^t e^{\lambda s} dw_{l_s} - e^{\lambda t} \phi(t) \right|^p dt \\ &\leq \sigma(2m+2)^{p-1} \left[\sum_{j=1}^m I_j + \sum_{j=1}^m |e^{\lambda \tau_j} \Delta w_j - \Delta a_j|^p + |\Delta a_m|^p + \left| \int_{\tau_m}^T e^{\lambda s} dw_{l_s} \right|^p \right]. \end{aligned} \quad (4.18)$$

Hence, by (4.12), (4.15), (4.17) and (4.18), we have

$$\begin{aligned} & \int_0^T \left| \int_0^t e^{\lambda s} dw_{l_s} - e^{\lambda t} \phi(t) \right|^p dt \\ & \leq 6^p T(m+1)^p \left[\sum_{j=1}^m I_j + \sum_{j=1}^m |e^{\lambda \tau_j} \Delta w_j - \Delta a_j|^p \right] \\ & \quad + \sigma 6^p(m+1)^p \sum_{k=1}^m |\Delta a_k|^p + \sigma 2^{p-1}(m+1)^{p-1} \left| \int_{\tau_m}^T e^{\lambda s} dw_{l_s} \right|^p, \end{aligned}$$

this, together with (4.11), immediately gives

$$\begin{aligned} & \int_0^T \left| \int_0^t e^{\lambda s} dw_{l_s} - e^{\lambda t} \phi(t) \right|^p dt + \left| \int_0^T e^{\lambda s} dw_{l_s} - e^{\lambda T} a \right| \\ & \leq 6^p T(m+1)^p \left[\sum_{j=1}^m (I_j + I_j^{1/p}) + \sum_{j=1}^m \left(|e^{\lambda \tau_j} \Delta w_j - \Delta a_j|^p + |e^{\lambda \tau_j} \Delta w_j - \Delta a_j| \right) \right] \\ & \quad + \sigma 6^p(m+1)^p \sum_{k=1}^m |\Delta a_k| + \left[\sigma 2^{p-1}(m+1)^{p-1} \left| \int_{\tau_m}^T e^{\lambda s} dw_{l_s} \right|^p + \left| \int_{\tau_m}^T e^{\lambda s} dw_{l_s} \right| \right]. \end{aligned} \quad (4.19)$$

For any $\varepsilon \in (0, 1)$, by the easy fact $\max_{1 \leq i \leq m} |\Delta a_i| \leq 2M$, choose $\sigma > 0$ sufficiently small, we have

$$\sigma 6^p(m+1)^p \sum_{k=1}^m |\Delta a_k| \leq \sigma 6^p(m+1)^p \cdot 2mM < \frac{\varepsilon}{2}.$$

Therefore the event

$$\left\{ \int_0^T \left| \int_0^t e^{\lambda s} dw_{l_s} - e^{\lambda t} \phi(t) \right|^p dt + |e^{\lambda T} z(T) - e^{\lambda T} a| \leq \varepsilon \right\} \subset \bigcap_{i=1}^m (A_i(l) \cap B_i(l)),$$

with

$$\begin{aligned} A_i(l) &:= \left\{ I_i + I_i^{1/p} \leq \frac{\varepsilon}{8 \cdot 6^p T(m+1)^p} \right\}, \\ B_i(l) &:= \left\{ |e^{\lambda \tau_i} \Delta w_i - \Delta a_i|^p + |e^{\lambda \tau_i} \Delta w_i - \Delta a_i| \leq \frac{\varepsilon}{8 \cdot 6^p T(m+1)^p} \right\}, \\ C(l) &:= \left\{ \left| \int_{\tau_m}^T e^{\lambda s} dw_{l_s} \right|^p + \left| \int_{\tau_m}^T e^{\lambda s} dw_{l_s} \right| < \frac{\varepsilon}{8 \cdot 2^{p-1}(m+1)^{p-1}T} \right\}, \end{aligned}$$

for all $i = 1, \dots, m$. Given the subordinator $S = l$, $A_1(l), B_1(l), \dots, A_m(l), B_m(l), C(l)$ are independent. By the reflection property of Brownian motion (see Proposition 3.3.7 in [23]), it is easy to calculate that for all $l \in \Delta_{t_0, \dots, t_m, \sigma, \delta}$, $i = 1, 2, \dots, m$, we have

$$\mathbb{P}^{\mathbb{W}}(A_i(l)) > 0, \quad \mathbb{P}^{\mathbb{W}}(B_i(l)) > 0, \quad \mathbb{P}^{\mathbb{W}}(C(l)) > 0,$$

while the last inequality can also be obtained by Eq. (4.3). Thus,

$$\begin{aligned} & \mathbb{E}^{\mathbb{S}} \left[\mathbb{P}^{\mathbb{W}} \left(\int_0^T |z(t) - \phi(t)|^p dt + |z(T) - a| \leq \varepsilon \right) \Big|_{l=S}, S \in \Delta_{t_0, \dots, t_m, \sigma, \delta} \right] \\ & \geq \mathbb{E}^{\mathbb{S}} \left\{ \mathbb{P}^{\mathbb{W}} [\cap_{i=1}^m (A_i(l) \cap B_i(l)) \cap C(l)] \Big|_{l=S}, S \in \Delta_{t_0, \dots, t_m, \sigma, \delta} \right\} \end{aligned}$$

$$= \mathbb{E}^{\mathbb{S}} \left\{ \prod_{i=1}^m \mathbb{P}^{\mathbb{W}}(A_i(l)) \mathbb{P}^{\mathbb{W}}(B_i(l)) \mathbb{P}(C(l)) \Big|_{l=S}, S \in \Delta_{t_0, \dots, t_m, \sigma, \delta} \right\}.$$

Which, together with (4.9), immediately implies

$$\mathbb{E}^{\mathbb{S}} \left[\mathbb{P}^{\mathbb{W}} \left(\int_0^T |z(t) - \phi(t)|^p dt + |z(T) - a| \leq \varepsilon \right) \Big|_{l=S}, S \in \Delta_{t_0, \dots, t_m, \sigma, \delta} \right] > 0. \quad (4.20)$$

Since $\{S \in \Delta_{t_0, \dots, t_m, \sigma, \delta}\} \subset \{S_T \leq T^2\}$, we immediately get (4.7), as desired.

The proof is complete. \blacksquare

4.2 Irreducibility in \mathbb{H}

Consider the deterministic system in \mathbb{H} ,

$$\partial_t x(t) + Ax(t) = N(x(t)) + u(t), \quad x(0) = x_0, \quad (4.21)$$

where $u \in L^2([0, T]; \mathbb{V})$. By using the similar argument in the proof of Lemma 4.2 in [29], for every $x(0) = x_0 \in \mathbb{H}$, $u \in L^2([0, T]; \mathbb{V})$, the system (4.21) admits a unique solution $x(\cdot) \in C([0, T]; \mathbb{H}) \cap C((0, T]; \mathbb{V})$. Moreover, $\{x(t)\}_{t \in [0, T]}$ has the following form:

$$x(t) = e^{-At} x_0 + \int_0^t e^{-A(t-s)} N(x(s)) ds + \int_0^t e^{-A(t-s)} u(s) ds, \quad \forall t \in [0, T]. \quad (4.22)$$

In [26], the following control problem of the deterministic system is proved.

Lemma 4.4 [26, Lemma 3.3] *For any $T > 0, \varepsilon > 0, a \in \mathbb{V}$, there exists some $u \in L^\infty([0, T]; \mathbb{V})$ such that the system (4.21) satisfies that*

$$\|x(T) - a\|_{\mathbb{V}} < \varepsilon.$$

Now we prove Theorem 4.1 by following the idea in [22, Theorem 5.4]. This approach has been used in the proof of Theorem [26, Theorem 2.3]. For the convenience of reading, we give the proof here.

Proof:[Proof of Theorem 4.1] Since Eq. (2.1) admits a unique mild solution $X_\cdot \in \mathbb{D}([0, \infty); \mathbb{H}) \cap \mathbb{D}((0, \infty); \mathbb{V})$, for any $x_0 \in \mathbb{H}, t > 0$, we have $X_t^{x_0} \in \mathbb{V}$ a.s.. By the Markov property of X , for any $a \in \mathbb{H}, T > 0, \varepsilon > 0$,

$$\begin{aligned} \mathbb{P}(\|X_T^{x_0} - a\|_{\mathbb{H}} < \varepsilon) &= \int_{\mathbb{V}} \mathbb{P}(\|X_T^{x_0} - a\|_{\mathbb{H}} < \varepsilon | X_t^{x_0} = v) \mathbb{P}(X_t^{x_0} \in dv) \\ &= \int_{\mathbb{V}} \mathbb{P}(\|X_{T-t}^v - a\|_{\mathbb{H}} < \varepsilon) \mathbb{P}(X_t^{x_0} \in dv). \end{aligned}$$

To prove that

$$\mathbb{P}(\|X_T^{x_0} - a\|_{\mathbb{H}} < \varepsilon) > 0,$$

it is sufficient to prove that for any $T > 0$,

$$\mathbb{P}(\|X_T^{x_0} - a\|_{\mathbb{H}} < \varepsilon) > 0 \quad \text{for all } x_0 \in \mathbb{V}.$$

Next, we prove the theorem under the assumption of the initial value $x_0 \in \mathbb{V}$ in the following two steps.

Step 1. For any $a \in \mathbb{H}, \varepsilon > 0$, there exists some $\theta > 0$ such that $e^{-\theta A} a \in \mathbb{V}$ and

$$\|a - e^{-\theta A} a\|_{\mathbb{H}} \leq \frac{\varepsilon}{4}. \quad (4.23)$$

For any $T > 0$, by Lemma 4.4 and the spectral gap inequality, there exists some $u \in L^\infty([0, T]; \mathbb{V})$ such that the system

$$\dot{x} + Ax = N(x) + u, \quad x(0) = x_0,$$

satisfies that

$$\|x(T) - e^{-\theta A}a\|_{\mathbb{H}} \leq \|x(T) - e^{-\theta A}a\|_{\mathbb{V}} < \frac{\varepsilon}{4}. \quad (4.24)$$

Putting (4.23) and (4.24) together, we have

$$\|x(T) - a\|_{\mathbb{H}} < \frac{\varepsilon}{2}. \quad (4.25)$$

Step 2: We shall consider the systems (4.26) and (4.27) as follows:

$$\begin{cases} \dot{z} + Az = u, & z(0) = 0, \\ \dot{y} + Ay = N(y + z), & y(0) = x_0 \in \mathbb{V}, \end{cases} \quad (4.26)$$

and

$$\begin{cases} dZ_t + AZ_t dt = Q_\beta dL_t, & Z_0 = 0; \\ dY_t + AY_t dt = N(Y_t + Z_t) dt, & Y_0 = x_0 \in \mathbb{V}. \end{cases} \quad (4.27)$$

By the arguments in the proof of Lemma 4.2 in [29], for any $x_0 \in \mathbb{V}, u \in L^2([0, T]; \mathbb{V})$, the systems (4.26) and (4.27) admit the unique solutions $(y(\cdot), z(\cdot)) \in C([0, T]; \mathbb{V})^2$ and $(Y, Z) \in C([0, T]; \mathbb{V}) \times \mathbb{D}([0, T]; \mathbb{V})$, a.s. Furthermore, denote

$$x(t) = y(t) + z(t), \quad X_t = Y_t + Z_t, \quad \forall t \geq 0.$$

For any $0 \leq t \leq T$,

$$\begin{aligned} & \|Y_t - y(t)\|_{\mathbb{H}}^2 + 2 \int_0^t \|Y_s - y(s)\|_{\mathbb{V}}^2 ds \\ &= 2 \int_0^t \langle Y_s - y(s), N(X_s) - N(x(s)) \rangle_{\mathbb{H}} ds \\ &= 2 \int_0^t \|Y_s - y(s)\|_{\mathbb{H}}^2 ds + 2 \int_0^t \langle Y_s - y(s), Z_s - z(s) \rangle_{\mathbb{H}} ds \\ & \quad - 2 \int_0^t \langle Y_s - y(s), X_s^3 - x^3(s) \rangle_{\mathbb{H}} ds. \end{aligned}$$

Let us estimate the third term of the right hand side. Denoting $\Delta Y_s = Y_s - y(s)$ and $\Delta Z_s = Z_s - z(s)$, we have

$$\begin{aligned} & \int_0^t \langle Y_s - y(s), X_s^3 - x^3(s) \rangle_{\mathbb{H}} ds \\ &= \int_0^t \langle \Delta Y_s, [\Delta Y_s + \Delta Z_s + x(s)]^3 - x^3(s) \rangle_{\mathbb{H}} ds \\ &= \int_0^t \langle \Delta Y_s, [\Delta Y_s + \Delta Z_s]^3 + 3[\Delta Y_s + \Delta Z_s]^2 x(s) + 3[\Delta Y_s + \Delta Z_s] x^2(s) \rangle_{\mathbb{H}} ds \\ &= \int_0^t \langle \Delta Y_s, (\Delta Y_s)^3 + 3(\Delta Y_s)^2 \Delta Z_s + 3\Delta Y_s (\Delta Z_s)^2 + (\Delta Z_s)^3 \rangle_{\mathbb{H}} ds \\ & \quad + 3 \int_0^t \langle \Delta Y_s, [(\Delta Y_s)^2 + 2\Delta Y_s \Delta Z_s + (\Delta Z_s)^2] x(s) \rangle_{\mathbb{H}} ds + 3 \int_0^t \langle \Delta Y_s, [\Delta Y_s + \Delta Z_s] x^2(s) \rangle_{\mathbb{H}} ds. \end{aligned}$$

Since $\frac{3}{4}(\Delta Y_s)^4 + 3(\Delta Y_s)^3 x(s) + 3(\Delta Y_s)^2 x^2(s) \geq 0$, from the above relation we have

$$\int_0^t \langle Y_s - y(s), X_s^3 - x^3(s) \rangle_{\mathbb{H}} ds$$

$$\begin{aligned}
&\geq \int_0^t \langle \Delta Y_s, 3(\Delta Y_s)^2 \Delta Z_s + 3\Delta Y_s (\Delta Z_s)^2 + (\Delta Z_s)^3 \rangle_{\mathbb{H}} ds \\
&\quad + 3 \int_0^t \langle \Delta Y_s, [2\Delta Y_s \Delta Z_s + (\Delta Z_s)^2] x(s) \rangle_{\mathbb{H}} ds \\
&\quad + 3 \int_0^t \langle \Delta Y_s, \Delta Z_s x^2(s) \rangle_{\mathbb{H}} ds + \frac{1}{4} \int_0^t \|\Delta Y_s\|_{L^4}^4 ds.
\end{aligned}$$

Using the following Young inequalities: for all $y, z \in L^4(\mathbb{T}; \mathbb{R})$,

$$\begin{aligned}
|\langle y, z \rangle_{\mathbb{H}}| &= \left| \int_{\mathbb{T}} y(\xi) z(\xi) d\xi \right| \leq \frac{\int_{\mathbb{T}} y^4(\xi) d\xi}{80} + C \int_{\mathbb{T}} z^{\frac{4}{3}}(\xi) d\xi, \\
|\langle y^2, z \rangle_{\mathbb{H}}| &= \left| \int_{\mathbb{T}} y^2(\xi) z(\xi) d\xi \right| \leq \frac{\int_{\mathbb{T}} y^4(\xi) d\xi}{80} + C \int_{\mathbb{T}} z^2(\xi) d\xi, \\
|\langle y^3, z \rangle_{\mathbb{H}}| &= \left| \int_{\mathbb{T}} y^3(\xi) z(\xi) d\xi \right| \leq \frac{\int_{\mathbb{T}} y^4(\xi) d\xi}{80} + C \int_{\mathbb{T}} z^4(\xi) d\xi,
\end{aligned} \tag{4.28}$$

and the Hölder inequality, we further get

$$\begin{aligned}
&\int_0^t \langle Y_s - y(s), X_s^3 - x^3(s) \rangle_{\mathbb{H}} ds \\
&\geq \frac{1}{80} \int_0^t \|\Delta Y_s\|_{L^4}^4 ds - 7C \int_0^t \|\Delta Z_s\|_{L^4}^4 ds \\
&\quad - 6C \int_0^t \|\Delta Z_s x(s)\|_{L^2}^2 ds - 3C \int_0^t \|(\Delta Z_s)^2 x(s)\|_{L^{\frac{4}{3}}}^{\frac{4}{3}} ds \\
&\quad - 3C \int_0^t \|\Delta Z_s x^2(s)\|_{L^{\frac{4}{3}}}^{\frac{4}{3}} ds \\
&\geq \frac{1}{80} \int_0^t \|\Delta Y_s\|_{L^4}^4 ds - 7C \int_0^t \|\Delta Z_s\|_{L^4}^4 ds \\
&\quad - 6C \int_0^t \|\Delta Z_s\|_{L^4}^2 \|x(s)\|_{L^4}^2 ds - 3C \int_0^t \|\Delta Z_s\|_{L^4}^{\frac{8}{3}} \|x(s)\|_{L^4}^{\frac{4}{3}} ds \\
&\quad - 3C \int_0^t \|\Delta Z_s\|_{L^4}^{\frac{4}{3}} \|x(s)\|_{L^4}^{\frac{8}{3}} ds.
\end{aligned}$$

Since $x(t) = y(t) + z(t) \in C([0, T]; \mathbb{V})$, by (3.3), there exists a constant C_T such that

$$\sup_{s \in [0, T]} \|y(s) + z(s)\|_{L^4} \leq \sup_{s \in [0, T]} \|y(s) + z(s)\|_{\mathbb{H}}^{\frac{1}{2}} \cdot \|y(s) + z(s)\|_{\mathbb{V}}^{\frac{1}{2}} \leq C_T.$$

Consequently, there is some constant $C_T > 0$ satisfying that

$$\begin{aligned}
&\|Y_t - y(t)\|_{\mathbb{H}}^2 + 2 \int_0^t \|Y_s - y(s)\|_{\mathbb{V}}^2 ds \\
&\leq 3 \int_0^t \|Y_s - y(s)\|_{\mathbb{H}}^2 ds + \int_0^t \|Z_s - z(s)\|_{\mathbb{H}}^2 ds \\
&\quad + C_T \int_0^t \left(\|Z_s - z(s)\|_{L^4}^4 + \|Z_s - z(s)\|_{L^4}^2 + \|Z_s - z(s)\|_{L^4}^{\frac{8}{3}} + \|Z_s - z(s)\|_{L^4}^{\frac{4}{3}} \right) ds.
\end{aligned}$$

Therefore, by the spectral gap inequality and Gronwall's inequality, we have

$$\|Y_T - y(T)\|_{\mathbb{H}}^2 \leq C_T \sum_{i \in \Lambda} \int_0^T \|Z_s - z(s)\|_{\mathbb{V}}^i ds, \tag{4.29}$$

where $\Lambda := \{4/3, 2, 8/3, 4\}$. This inequality, together with Lemma 4.3, (4.25), implies

$$\begin{aligned}
& \mathbb{P}(\|X_T - a\|_{\mathbb{H}} < \varepsilon) \\
&= \mathbb{P}(\|Y_T - y(T) + Z_T - z(T) + x(T) - a\|_{\mathbb{H}} < \varepsilon) \\
&\geq \mathbb{P}(\|Y_T - y(T)\|_{\mathbb{H}} \leq \varepsilon/4, \|Z_T - z(T)\|_{\mathbb{H}} \leq \varepsilon/4, \|x(T) - a\|_{\mathbb{H}} < \varepsilon/2) \\
&= \mathbb{P}(\|Y_T - y(T)\|_{\mathbb{H}} \leq \varepsilon/4, \|Z_T - z(T)\|_{\mathbb{H}} \leq \varepsilon/4) \\
&\geq \mathbb{P}\left(\sum_{i \in \Lambda} \int_0^T \|Z_s - z(s)\|_{\mathbb{V}}^i ds + \|Z_T - z(T)\|_{\mathbb{V}} \leq C_{T,\varepsilon}\right) \\
&> 0.
\end{aligned}$$

The proof is complete. ■

5 LDP for the occupation time

5.1 LDP for the occupation time

In this section, we recall some general results on the LDP for strong Feller and irreducible Markov processes. We follow [28].

Let E be a Polish metric space. Consider a general E -valued càdlàg Markov process

$$(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{F}, \{X_t(\omega)\}_{t \geq 0}, \{\mathbb{P}_x\}_{x \in E}),$$

where

- $\Omega = D([0, +\infty); E)$, which is the space of the càdlàg functions from $[0, +\infty)$ to E equipped with the Skorokhod topology; for any $\omega \in \Omega$, $X_t(\omega) = \omega(t)$;
- $\mathcal{F}_t^0 = \sigma\{X_s; 0 \leq s \leq t\}$ for any $t \geq 0$ (nature filtration);
- $\mathcal{F} = \sigma\{X_t; t \geq 0\}$ and $\mathbb{P}_x(X_0 = x) = 1$.

Hence, \mathbb{P}_x is the law of the Markov process with initial state $x \in E$. For any initial measure ν on E , let $\mathbb{P}_\nu(d\omega) := \int_E \mathbb{P}_x(d\omega) \nu(dx)$. Its transition probability is denoted by $\{P_t(x, dy)\}_{t \geq 0}$.

For all $f \in b\mathcal{B}(E)$, define

$$P_t f(x) = \int_E P_t(x, dy) f(y) \quad \text{for all } t \geq 0, x \in E.$$

$\{P_t\}_{t \geq 0}$ is *accessible* to $x \in E$, if the resolvent $\{\mathcal{R}_\lambda\}_{\lambda > 0}$ satisfies

$$\mathcal{R}_\lambda(y, \mathcal{U}) := \int_0^\infty e^{-\lambda t} P_t(y, \mathcal{U}) dt > 0, \quad \forall \lambda > 0$$

for all $y \in E$ and all neighborhoods \mathcal{U} of x . Notice that the accessibility of $\{P_t\}_{t \geq 0}$ to any $x \in E$ is the so called *topological transitivity* in Wu [28].

The empirical measure of level-3 (or process level) is given by

$$R_t := \frac{1}{t} \int_0^t \delta_{\theta_s X} ds$$

where $(\theta_s X)_t = X_{s+t}$ for all $t, s \geq 0$ are the shifts on Ω . Thus, R_t is a random element of $\mathcal{M}_1(\Omega)$, the space of all probability measures on Ω .

The level-3 entropy functional of Donsker-Varadhan $H : \mathcal{M}_1(\Omega) \rightarrow [0, +\infty]$ is defined by

$$H(Q) := \begin{cases} \mathbb{E}^{\bar{Q}} h_{\mathcal{F}_1^0}(\bar{Q}_{w(-\infty, 0]}; \mathbb{P}_{w(0)}) & \text{if } Q \in \mathcal{M}_1^s(\Omega); \\ +\infty & \text{otherwise,} \end{cases}$$

where

- $\mathcal{M}_1^s(\Omega)$ is the subspace of $\mathcal{M}_1(\Omega)$, whose elements are moreover stationary;
- \bar{Q} is the unique stationary extension of $Q \in \mathcal{M}_1^s(\Omega)$ to $\bar{\Omega} := D(\mathbb{R}; E)$; $\mathcal{F}_t^s = \sigma\{X(u); s \leq u \leq t\}$, $\forall s, t \in \mathbb{R}, s \leq t$;
- $\bar{Q}_{w(-\infty, t]}$ is the regular conditional distribution of \bar{Q} knowing $\mathcal{F}_t^{-\infty}$;
- $h_{\mathcal{G}}(\nu; \mu)$ is the usual relative entropy or Kullback information of ν with respect to μ restricted to the σ -field \mathcal{G} , given by

$$h_{\mathcal{G}}(\nu; \mu) := \begin{cases} \int \frac{d\nu}{d\mu}|_{\mathcal{G}} \log \left(\frac{d\nu}{d\mu}|_{\mathcal{G}} \right) d\mu & \text{if } \nu \ll \mu \text{ on } \mathcal{G}; \\ +\infty & \text{otherwise.} \end{cases}$$

The level-2 entropy functional $J : \mathcal{M}_1(E) \rightarrow [0, \infty]$ which governs the LDP in our main result is

$$J(\mu) = \inf\{H(Q) | Q \in \mathcal{M}_1^s(\Omega) \text{ and } Q_0 = \mu\}, \quad \forall \mu \in \mathcal{M}_1(E), \quad (5.1)$$

where $Q_0(\cdot) = Q(X_0 \in \cdot)$ is the marginal law at $t = 0$.

5.1.1 The hyper-exponential recurrence criterion

Recall the following hyper-exponential recurrence criterion for LDP established by Wu [28, Theorem 2.1].

For any measurable set $K \in E$, let

$$\tau_K := \inf\{t \geq 0 \text{ s.t. } X_t \in K\}, \quad \tau_K^{(1)} := \inf\{t \geq 1 \text{ s.t. } X_t \in K\}. \quad (5.2)$$

Theorem 5.1 [28] *Let $\mathcal{A} \subset \mathcal{M}_1(E)$ and assume that*

$$\{P_t\}_{t \geq 0} \text{ is strong Feller and topologically irreducible on } E.$$

If for any $\lambda > 0$, there exists some compact set $K \subset\subset E$, such that

$$\sup_{\nu \in \mathcal{A}} \mathbb{E}^{\nu} e^{\lambda \tau_K} < \infty, \quad \text{and} \quad \sup_{x \in K} \mathbb{E}^x e^{\lambda \tau_K^{(1)}} < \infty. \quad (5.3)$$

Then the family $\mathbb{P}_{\nu}(\mathcal{L}_t \in \cdot)$ satisfies the LDP on $\mathcal{M}_1(E)$ w.r.t. the τ -topology with the rate function J defined by (5.1), and uniformly for initial measures ν in the subset \mathcal{A} . More precisely, the following three properties hold:

- (a1) *for any $a \geq 0$, $\{\mu \in \mathcal{M}_1(E); J(\mu) \leq a\}$ is compact in $(\mathcal{M}_1(E), \tau)$;*
- (a2) *(the lower bound) for any open set G in $(\mathcal{M}_1(E), \tau)$,*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \log \inf_{\nu \in \mathcal{A}} \mathbb{P}_{\nu}(\mathcal{L}_T \in G) \geq -\inf_G J;$$

- (a3) *(the upper bound) for any closed set F in $(\mathcal{M}_1(E), \tau)$,*

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log \sup_{\nu \in \mathcal{A}} \mathbb{P}_{\nu}(\mathcal{L}_T \in F) \leq -\inf_F J.$$

5.2 The proof of Theorem 2.3

In this section, we shall prove Theorem 2.3 according to Theorem 5.1.

Proof: [Proof of Theorem 2.3] Let $\{X_t\}_{t \geq 0}$ be the solution to Eq. (2.1) with initial value $x \in \mathbb{H}$. By Theorems 3.2 and 4.1, we know that X is strong Feller and irreducible in H . According to Theorem 5.1, to prove Theorem 2.3, we need prove that the hyper-exponential recurrence condition 5.3 is fulfilled. The verification of this condition will be given by Theorem 5.5 below. \blacksquare

Let $Y_t := X_t - Z_t$. Then Y_t satisfies the following equation:

$$dY_t + AY_t dt = N(Y_t + Z_t) dt, \quad Y_0 = x. \quad (5.4)$$

Lemma 5.2 *For all $T > 0$, we have*

$$\sup_{t \in [T/2, T]} \|Y_t\|_{\mathbb{H}} \leq C(T) \left(1 + \sup_{0 \leq t \leq T} \|Z_t\|_{\mathbb{V}} \right), \quad (5.5)$$

where the constant $C(T)$ does not depend on the initial value $Y_0 = x$.

Proof: By the chain rule, we obtain that

$$\frac{d\|Y_t\|_{\mathbb{H}}^2}{dt} + 2\|Y_t\|_{\mathbb{V}}^2 = 2\langle Y_t, N(Y_t + Z_t) \rangle. \quad (5.6)$$

Using the Young inequalities (4.28), Hölder inequality and the elementary inequality $2\sqrt{a} \leq a/b + b$ for all $a, b > 0$, we obtain that there exists a constant $C \geq 1$ satisfying that

$$2\langle Y_t, N(Y_t + Z_t) \rangle \leq -\|Y_t\|_{L^4}^4 + C(1 + \|Z_t\|_{L^4}^4).$$

This inequality, together with Eq. (3.3), Eq. (5.6) and Hölder inequality, implies that

$$\frac{d\|Y_t\|_{\mathbb{H}}^2}{dt} + 2\|Y_t\|_{\mathbb{V}}^2 \leq -\|Y_t\|_{\mathbb{H}}^4 + C(1 + \|Z_t\|_{\mathbb{V}}^4). \quad (5.7)$$

For any $t \geq 0$, denote

$$h(t) := \|Y_t\|_{\mathbb{H}}^2, \quad K_T := \sup_{0 \leq t \leq T} \sqrt{C(1 + \|Z_t\|_{\mathbb{V}}^4)} \geq 1.$$

By Eq. (5.7), we have

$$\frac{dh(t)}{dt} \leq -h^2(t) + K_T^2, \quad \forall t \in [0, T],$$

with the initial value $h(0) = \|x\|_{\mathbb{H}}^2 \geq 0$.

By the comparison theorem (e.g., the deterministic case of [15, Chapter VI, Theorem 1.1]), we obtain that

$$h(t) \leq g(t), \quad \forall t \in [0, T], \quad (5.8)$$

where the function g solves the following equation

$$\frac{dg(t)}{dt} = -g^2(t) + K_T^2, \quad \forall t \in [0, T], \quad (5.9)$$

with the initial value $g(0) = h(0)$. The solution of Eq. (5.9) is

$$g(t) = K_T + 2K_T \left(\frac{g(0) + K_T}{g(0) - K_T} e^{2K_T t} - 1 \right)^{-1}, \quad \forall t \in [0, T],$$

where it is understood that $g(t) \equiv K_T$ when $g(0) = K_T$. It is easy to show that for any initial value $g(0)$, we have

$$g(t) \leq K_T (1 + 2(e^T - 1)^{-1}), \quad \forall t \in [T/2, T].$$

This inequality, together with Eq. (5.8) and the definition of K_T , immediately implies the required estimate (5.5).

The proof is complete. \blacksquare

Lemma 5.3 For all $T \geq 1$, $\delta \in (0, 1)$ and $p \in (0, \alpha/4)$, we have

$$\mathbb{E}^x [\|Y_T\|_\delta^p] \leq C_{\alpha,p} T,$$

where the constant $C_{\delta,p}$ does not depend on the initial value $Y_0 = x$ and T .

Proof: Since

$$Y_T = e^{-AT/2} Y_{T/2} + \int_{T/2}^T e^{-A(T-s)} N(Y_s + Z_s) ds,$$

for any $\delta \in (0, 1)$, by the inequalities (3.2)-(3.9) and Lemma 5.2, there exists a constant $C = C_{T,\delta}$ (whose value may be different from line to line by convention) satisfied that

$$\begin{aligned} \|Y_T\|_\delta &\leq C \|Y_{T/2}\|_{\mathbb{H}} + C \int_{T/2}^T (T-s)^{-\frac{\delta}{2}} \|N(Y_s + Z_s)\|_{\mathbb{H}} ds \\ &\leq C \|Y_{T/2}\|_{\mathbb{H}} + C \int_{T/2}^T (T-s)^{-\frac{\delta}{2}} (\|Y_s\|_{\mathbb{H}} + \|Z_s\|_{\mathbb{H}} + \|Y_s^3\|_{\mathbb{H}} + \|Z_s^3\|_{\mathbb{H}}) ds \\ &\leq C \|Y_{T/2}\|_{\mathbb{H}} + C \int_{T/2}^T (T-s)^{-\frac{\delta}{2}} (\|Y_s\|_{\mathbb{H}} + \|Z_s\|_{\mathbb{V}} + \|Y_s\|_{\mathbb{V}} \cdot \|Y_s\|_{\mathbb{H}}^2 + \|Z_s\|_{\mathbb{V}}^3) ds \\ &\leq C \left(1 + \sup_{0 \leq t \leq T} \|Z_t\|_{\mathbb{V}}^3\right) + C \int_{T/2}^T (T-s)^{-\frac{\delta}{2}} \|Y_s\|_{\mathbb{V}} \cdot \|Y_s\|_{\mathbb{H}}^2 ds. \end{aligned}$$

Next, we estimate the last term in above inequality: by Eq. (5.7) and Lemma 5.2 again, we have

$$\begin{aligned} &\int_{T/2}^T (T-s)^{-\frac{\delta}{2}} \|Y_s\|_{\mathbb{V}} \|Y_s\|_{\mathbb{H}}^2 ds \\ &\leq C \left(1 + \sup_{0 \leq t \leq T} \|Z_t\|_{\mathbb{V}}^2\right) \int_{T/2}^T (T-s)^{-\frac{\delta}{2}} \|Y_s\|_{\mathbb{V}} ds \\ &\leq C \left(1 + \sup_{0 \leq t \leq T} \|Z_t\|_{\mathbb{V}}^2\right) \left(\int_{T/2}^T (T-s)^{-\delta} ds\right)^{\frac{1}{2}} \left(\int_{T/2}^T \|Y_s\|_{\mathbb{V}}^2 ds\right)^{\frac{1}{2}} \\ &\leq C \left(1 + \sup_{0 \leq t \leq T} \|Z_t\|_{\mathbb{V}}^2\right) \left(\|Y_{T/2}\|_{\mathbb{H}}^2 + \int_{T/2}^T (1 + \|Z_s\|_{\mathbb{V}}^4) ds\right)^{\frac{1}{2}} \\ &\leq C \left(1 + \sup_{0 \leq t \leq T} \|Z_t\|_{\mathbb{V}}^4\right). \end{aligned}$$

Hence, by Lemma 3.1 (taking $\gamma = 1$ there), we obtain that for any $p \in (0, (1+\delta)\alpha/8)$,

$$\begin{aligned} \mathbb{E}^x [\|Y_T\|_\delta^p] &\leq C \left(1 + \mathbb{E}^x \left[\sup_{0 \leq t \leq T} \|Z_t\|_{\mathbb{V}}^{4p} \right]\right) \\ &\leq C_{\alpha,p} T^{\frac{p}{\alpha}} \left(1 + T^{\frac{1-p}{2}}\right) \leq 2C_{\alpha,p} T, \end{aligned}$$

where $C_{\delta,p}$ is independent of x and T .

The proof is complete. ■

By Lemma 3.1 and Lemma 5.2, we can get the following estimate.

Theorem 5.4 For all $T > 0$, $\delta \in (0, 1)$ and $p \in (0, (1+\delta)\alpha/8)$, we have

$$\mathbb{E}^x [\|X_T\|_\delta^p] \leq C_{\delta,p} T,$$

where the constant $C_{\delta,p}$ does not depend on the initial value $X_0 = x$ and T . Consequently, the Markov property, it follows from the Markov property that

$$\sup_{t \geq T} \mathbb{E}^x [\|X_t\|_\delta^p] \leq C_{\delta,p} T.$$

5.2.1 The hyper-exponential Recurrence

In this part, we will verify the hyper-exponential recurrence condition (5.3).

For any $\delta \in (0, 1)$, $M > 0$, define the hitting time of $\{X_n\}_{n \geq 1}$:

$$\tau_M := \inf\{k \geq 1 : \|X_k\|_\delta \leq M\}. \quad (5.10)$$

Let

$$K := \{x \in \mathbb{H} : \|x\|_\delta \leq M\}.$$

Clearly, K is compact in \mathbb{H} . Recall the definitions of τ_K and $\tau_K^{(1)}$ in (5.2). It is obvious that

$$\tau_K \leq \tau_M, \quad \tau_K^{(1)} \leq \tau_M. \quad (5.11)$$

This fact, together with the following important theorem, implies the hyper-exponential recurrence condition (5.3).

Theorem 5.5 *For any $\lambda > 0$, there exists $M = M_{\lambda, \delta}$ such that*

$$\sup_{\nu \in \mathcal{M}_1(\mathbb{H})} \mathbb{E}^\nu[e^{\lambda \tau_M}] < \infty.$$

Proof: For any $n \in \mathbb{N}$, let

$$B_n := \{\|X_j\|_\delta > M; j = 1, \dots, n\} = \{\tau_M > n\}.$$

By the Markov property of $\{X_n\}_{n \in \mathbb{N}}$, Chebychev's inequality and Theorem 5.4, we obtain that for any $\nu \in \mathcal{M}_1(\mathbb{H})$, $p \in (0, \alpha/4)$,

$$\begin{aligned} \mathbb{P}_\nu(B_n) &= \mathbb{P}_\nu(B_{n-1}) \cdot \mathbb{P}_\nu(B_n | B_{n-1}) \\ &\leq \mathbb{P}_\nu(B_{n-1}) \cdot \frac{\mathbb{E}_{X_{n-1}}[\|X_n\|_\delta^p]}{M^p} \\ &\leq \mathbb{P}_\nu(B_{n-1}) \cdot \frac{C_{\delta, p}}{M^p}, \end{aligned}$$

where $C_{\delta, p}$ is the constant in Lemma 5.4 (taking $T = 1$).

By induction, we have for any $n \geq 0$,

$$\mathbb{P}_\nu(\tau_M > n) = \mathbb{P}_\nu(B_n) \leq \left(\frac{C_{\delta, p}}{M^p} \right)^n.$$

This inequality, together with Fubini's theorem, implies that for any $\lambda > 0, \nu \in \mathcal{M}_1(\mathbb{H})$,

$$\begin{aligned} \mathbb{E}_\nu[e^{\lambda \tau_M}] &= \int_0^\infty \lambda e^{\lambda t} \mathbb{P}_\nu(\tau_M > t) dt \\ &\leq \sum_{n=0}^\infty \lambda e^{\lambda(n+1)} \mathbb{P}_\nu(\tau_M > n) \\ &\leq \sum_{n=0}^\infty \lambda e^{\lambda(n+1)} \left(\frac{C_{\delta, p}}{M^p} \right)^n, \end{aligned}$$

which is finite as $M > (C_{\delta, p} e^\lambda)^{1/p}$.

The proof is complete. ■

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